ARCH type bilinear weakly dependent models

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Weak dependence for infinite $ARCH$-type bilinear models

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Giraitis and Surgailis (2002) introduced $ARCH$-type bilinear models for their specific long range dependence properties. We rather consider weak dependence properties of these models. The computation of mixing coefficients for such models does not look as an accessible objective. So, we resort to the notion of weak dependence introduced by Doukhan and Louhichi (1999), whose use seems more relevant here. The decay rate of the weak dependence coefficients sequence is established under different specifications of the model coefficients. This implies various limit theorems and asymptotics for statistical procedures. We also derive bounds for the joint densities of this model in the case of regular inputs.
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1 Introduction and motivations

1.1 *Infinite ARCH-type bilinear models*

A vast literature is devoted to the study of conditionally heteroskedastic models. One of the best-known model is the $GARCH$ model (Generalized Autoregressive Conditionally Heteroskedastic) introduced by Engle [17] and Bollerslev [6]. A usual $GARCH(p,q)$ model can be written

$$r_t = \sigma_t \xi_t, \quad \sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 + \sum_{j=1}^{q} \alpha_j r_{t-j}^2$$

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where $\alpha_0 \geq 0,\ \beta_i \geq 0,\ \alpha_j \geq 0,\ p \geq 0,\ q \geq 0$ are the model parameters and the $(\xi_t)_t$ are independent and identically distributed (iid). If the β_i are null, we have an $ARCH(q)$ model which can be extended in $ARCH(\infty)$ model, see Robinson [26], Giraitis and Robinson [20], Robinson and Zaffaroni [27], Kokozska and Leipus [23], Kazakevicius and Leipus [22]. These models are often used in finance because their properties are close to the properties observed on empirical financial data such as heavy tails, volatility clustering, white noise behaviour or autocorrelation of the squared series. To reproduce other properties of the empirical data such as leverage effect, a lot of extensions of the $GARCH$ model have been introduced as $EGARCH$ or $TGARCH$, see Zakoïan [28], El babsiri and Zakoïan [16].

In this paper, we study weak dependence properties of $ARCH$-type bilinear models introduced by Giraitis and Surgailis [21]. An $ARCH$-type bilinear model can be written

$$X_t = \varepsilon_t\Big(a + \sum_{j=1}^{\infty} a_j X_{t-j}\Big) + b_0 + \sum_{j=1}^{\infty} b_j X_{t-j}, \tag{1}$$

where a, (a_j) and (b_j) are real coefficients and the $(\varepsilon_t)_{t\in\mathbb{Z}}$ are iid centered. We usually consider $b_0 = 0$. $ARCH(\infty)$, $GARCH(p,q)$ and $LARCH$ models are particular cases of the bilinear models, see Giraitis, Leipus and Surgailis [19], Giraitis, Kokoszka, Leipus and Teyssière [18]. Quote that Doukhan, Teyssière and Winant [15] introduced a very general vector valued version of this model. Giraitis and Surgailis [21] prove that under restrictions, there is a unique stationary solution for these models. This solution has a chaotic expansion. The following assumption is necessary to define this solution.

Assumption $\mathcal{H}$. The $(\varepsilon_t)_{t\in\mathbb{Z}}$ are iid centered, $\mathbb{E}|\varepsilon_1| < \infty$ and the power series $A(z) = \sum_{j=1}^{\infty} a_j z^j$ and $B(z) = \sum_{j=1}^{\infty} b_j z^j$ exist for $|z| \leq 1$.

We define $\|\xi\|_{L^p} = (\mathbb{E}|\xi|^p)^{1/p}$ and $\|h\|_p^p = (\sum_{j=0}^{\infty} |h_j|^p)^{1/p}$, for $p \in [1,\infty)$, with usual extension to the supremum norm if $p = +\infty$. We set

$$G(z) = \{1 - B(z)\}^{-1} = \sum_{j=0}^{\infty} g_j z^j \text{ and } H(z) = A(z)\{1 - B(z)\}^{-1} = \sum_{j=1}^{\infty} h_j z^j.$$

Let $(a\star b)_j = \sum_{i=0}^{j} a_i b_{j-i}$ denote the convolution and $a_j^{(n)} = a_j \mathbf{1}(1 \leq j \leq n)$, where $\mathbf{1}$ is the indicator function. Giraitis and Surgailis established in [21] the following proposition

Proposition 1.1 **(Giraitis, Surgailis)** *Assume that the (ε_t) are iid, centered at expectation and such that $\|\varepsilon_t\|_2 = 1$. If $\|(a^{(n)} - a) \star g\|_2$ and $\|(b^{(n)} - b) \star g\|_2$ tend to zero as n goes to infinity and if $\|h\|_2 < 1$, $\|g\|_2 < \infty$, then there exists a solution of equation* (1) *which is unique, strictly stationary and given by*

$$X_t = a \sum_{k=1}^{\infty} \sum_{s_k < \cdots < s_1 \le t} g_{t-s_1} h_{s_1 - s_2} \cdots h_{s_{k-1} - s_k} \ \varepsilon_{s_1} \cdots \varepsilon_{s_k}. \tag{2}$$

1.2 *Weak dependence*

To our knowledge, there is no study of the weak dependence properties of $ARCH$ or $GARCH$ type models with infinite number of coefficients. In fact, the use of mixing coefficients is very technical and necessitates additional regularity assumptions, see Doukhan [10]. In the case of finite memory ARCH models, Mokkadem derives in [24] the absolute regularity properties of such models. An extension to infinite memory case seems quite doubtful because of Andrew's example of a non mixing first order autoregressive process, see [10]. We add that mixing conditions also necessitate some regularity properties of the innovation process. In order to derive limit theorems for functionals of such models, we prove in this paper that a causal version of the weak dependence property introduced by Doukhan and Louhichi in [12] holds. Indeed, such weak dependence conditions is a simple alternative to mixing. It also yields all kinds of limit theorems.

Let $f : \mathbb{R}^d \to \mathbb{R}$, we define

$$\mathrm{Lip}\,(f) = \sup_{(x_1,\ldots,x_d) \neq (y_1,\ldots,y_d)} \frac{|f(x_1,\ldots,x_d) - f(y_1,\ldots,y_d)|}{|x_1 - y_1| + \cdots + |x_d - y_d|}.$$

We recall the definition of $\theta-$weak dependence introduced by Doukhan and Louhichi [12] and Dedecker and Doukhan [7] :

Definition 1.2 $(X_n)_{n\in\mathbb{Z}}$ is $\theta-$weakly dependent if there is a sequence $(\theta_i)_i$ such that $\lim_i \theta_i = 0$ and $(X_n)_n$ satisfies

$$\left|\mathrm{Cov}\big[f\big(X_{i_1},\ldots,X_{i_u}\big), g\big(X_{j_1},\ldots,X_{j_v}\big)\big]\right| \le \theta_i v \|f\|_\infty \mathrm{Lip}\,(g)$$

for all u, v, $i_1 \le i_2 \le \cdots \le i_u \le i_u + i \le j_1 \le j_2 \le \cdots \le j_v$ and any measurable functions $f : \mathbb{R}^u \to \mathbb{R}$ and $g : \mathbb{R}^v \to \mathbb{R}$ such that $\|f\|_\infty, \mathrm{Lip}\,(g) < \infty$.

This condition implies limit theorems such as

– *Donsker invariance principle*, see Dedecker and Doukhan [7] :

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]} X_k \xrightarrow[n\to\infty]{D[0,1]} \sigma W_t,$$

where W_t is a standard Brownian motion and $\sigma^2 = \sum_{k=-\infty}^{\infty} \mathrm{Cov}(X_0, X_k) \geq 0$ is well defined, if for some positive δ,

$$\mathbb{E}|X_0|^{2+\delta} < \infty \text{ and } \sum_{i>0} i^{1/\delta}\theta_i < \infty.$$

– *Empirical central limit theorem*, see Prieur [25] :

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\{\mathbf{1}(X_k \leq x) - F(x)\} \xrightarrow[n\to\infty]{D[\bar{\mathbb{R}}]} Z(x),$$

where $\{Z(x)\}_{x\in\,\mathbb{R}}$ is the centered Gaussian process with covariance function

$$\Gamma(x,y) = \sum_{k=-\infty}^{\infty} \mathrm{Cov}[\mathbf{1}(X_0 \leq x), \mathbf{1}(X_k \leq y)],$$

if $\theta_i = \mathcal{O}(i^{-a})$ for $a > 2 + 2\sqrt{2}$ and the marginal distribution of X_0 is atomless.

– Other statistical asymptotic features are considered in [1], [2], [7], [9], [12] and [13]. See also section 3.

Under restrictions on the model coefficients, we derive explicit bounds for these weak dependence coefficients and thus directly obtain asymptotic results for this process.

We first establish sufficient conditions for the existence of the chaotic solution (2) in L^p. Indeed in proposition 1.1, the existence condition in L^2 is based on the coefficients (h_j). We prefer conditions on the coefficients (a_j) and (b_j). Thus we extend the result of existence of the chaotic expansion to L^p. After this, we establish bounds for the decay of the weak dependence coefficients sequence in different cases. We define under $\mathcal{H}$ the following assumption :

Assumption $\mathcal{H}'$. Let $\tilde{h} = \|h\|_1 \|\varepsilon_1\| \|_{L^1}$, $\tilde{h} < 1$ and $\|g\|_1 < \infty$.

Remark. Assumption $\mathcal{H}'$ is granted as soon as the coefficients (b_j) are non negative and $\|a\|_1 \|\varepsilon_1\|_{L^1} + \|b\|_1 < 1$, see section 4.1.

We now define our different specifications :

(a) *Markovian case* : $\mathcal{H}$, $\eta = \left(\sum_{j=1}^{J} \mathbb{E}|a_j\varepsilon_1 + b_j|\right)^{1/J} < 1$ with J such that

$$\forall j > J,\ a_j = b_j = 0.$$

(b) *Geometric decay* : $\mathcal{H}$, $\mathcal{H}'$ and

$$\exists \alpha \in]0,1[,\ \beta > 1 :\ \sum_j \beta^j |b_j| \leq 1 \text{ and } |a_j| \leq \alpha^j.$$

Remark. Our definition of the geometric case gets for particular case a more classical definition where we suppose there exists $0 < \zeta < 1$, and $0 < \lambda < \frac{1-\zeta}{\zeta}$ such that, for all $j \geq 1$ we have, $0 \leq b_j \leq \lambda \zeta^j$.

(c) *Riemannian decay* : $\mathcal{H}$, $\mathcal{H}'$, $\|b\|_1 < 1$ and

$$\exists \beta > 1,\ \alpha > 1 :\ \sum_j j^\beta |b_j| = B < \infty, \text{ and } \sum_j j^\alpha |a_j| = A < \infty.$$

In the following, we shall systematically refer to the previous conditions (a), (b) and (c).

Applications of our results are given in section 3. We also prove in section 3 that if all the coefficients are non negative, then each vector $(X_1, \ldots, X_n)$ admits a density conditional on the past of the process if this is the case for the innovations. We also show that we can uniformly control the density of any couple (X_1, X_i). Such results are very useful for functional estimation as stressed in [2], [13] and [3]. Section 4 contains the proofs.

2 Properties of bilinear models

2.1 *Existence of the solution in* $L^p(\Omega,\ \mathcal{A},\ \mathbb{P})$

THEOREM 2.1 *If the* (ε_t) *are iid and belong to* $L^p(\Omega,\ \mathcal{A},\ \mathbb{P})$,

(i) a sufficient condition of existence in $L^p(\Omega,\ \mathcal{A},\ \mathbb{P})$ *of expansion* (2) *is* $\|g_1\| < \infty$ *and* $\|h\|_1 \|\varepsilon_1\|_{L^p} < 1$.

(ii) The preceding condition is implied by the non negativity of the coefficients (b_j) *and the inequality* $\|a\|_1\|\varepsilon\|_{L^p} + \|b\|_1 < 1$.

Remark. The result extends to dependent innovations (ε_t). Existence in $L^1(\Omega, \mathcal{A}, \mathbb{P})$ of the chaotic expansion (2) holds if $\sup_t \|\varepsilon_t\|_{L^\infty} \le M$, $\|g_1\| < \infty$ and $M\|h\|_1 < 1$. A condition on the coefficients (a_j) and (b_j) is the following : if the (b_j) are non negative, $\sup_t \|\varepsilon_t\|_{L^\infty} \le M$ and $\|a\|_1 M + \|b\|_1 < 1$ then the expression (2) exists in $L^1(\Omega, \mathcal{A}, \mathbb{P})$.

2.2 *Weak dependence coefficients*

From now on, we assume that the solution exists in $L^1(\Omega, \mathcal{A}, \mathbb{P})$, is stationary and is given by equation (2).

THEOREM 2.2 **(Bounds on θ)**

(a) In the Markovian case,

$$\theta_r = \mathcal{O}(\eta^r) \text{ as } r \to \infty.$$

(b) Under geometric decay, for any β_1 *such that* $1 < \beta_1 < \beta$,

$$\theta_r = \mathcal{O}\big(e^{-c\sqrt{r}}\big) \text{ as } r \to \infty,$$

where $c = (\log \tilde{h} \log m)^{1/2}$ *with* $m = \alpha^{\frac{-\log \beta_1}{\log(\alpha/\beta_1)}}$.

(c) Under Riemannian decay,

$$\theta_r = \mathcal{O}\Big\{\Big(\frac{r}{\log r}\Big)^{-d}\Big\} \text{ as } r \to \infty,$$

where $d = \min\left(\frac{(\beta-1)\log\rho}{(1-\beta)\log 2+\log\rho}, \alpha\right)$ *with* $\rho = \frac{1}{1+\frac{1-\|b\|_1}{B}}$.

3 Applications

Theorem 2.2 leads to various applications in the bilinear context. We first precise below conditions to get Donsker invariance principle and empirical central limit theorem. A last subsection devoted to conditional densities is more specific to our models. The results of this section are particularly relevant for functional estimation. Notice also that theorem 2.2 enables to obtain exponential inequalities, see [14]. Results for stochastic algorithms, Whittle estimator, and copula can be respectively found in [5], [4] and [11]. For a general review of these properties, see [8].

3.1 *Donsker invariance principle*

Let $\mathcal{P}_\delta$ be the following condition : for some $\delta > 0, \mathbb{E}|X_0|^{2+\delta} < \infty$. Using the results of Dedecker and Doukhan [7] together with theorem 2.2, we easily obtained the Donsker invariance principle (see section 1.2) under the following assumptions

(a) *in the Markovian case* : $\mathcal{P}_\delta$ and $\eta < 1$,

(b) *under geometric decay* : $\mathcal{P}_\delta$,

(c) *under Riemannian decay* : $\mathcal{P}_\delta$ and $d > 1 + 1/\delta$ with d defined in theorem 2.2.

Note in particular that in the important case of $LARCH$ models ($b_j = 0$), in the Markovian case, the second condition is reduced to $\mathbb{E}|\varepsilon_1| \sum_{j=1}^{J} |a_j| < 1$ and under Riemannian decay to $\alpha > 1 + 1/\delta$.

3.2 *Empirical central limit theorem*

Let $\mathcal{P}'$ be the following condition : the marginal distribtion of X_0 is atomless. The results of Prieur together with theorem 2.2 enable us to get the empirical central limit theorem (see section 1.2) under the following assumptions

(i) *in the Markovian case* : $\mathcal{P}'$ and $\eta < 1$,

(ii) *under geometric decay* : $\mathcal{P}'$,

(iii) *under Riemannian decay* : $d > 2 + 2\sqrt{2}$ with d defined in theorem 2.2.

Note also that for $LARCH$ models, in the Markovian case the second condition is reduced to $\mathbb{E}|\varepsilon_1| \sum_{j=1}^{J} |a_j| < 1$ and under Riemannian decay to $\alpha > 2+2\sqrt{2}$.

3.3 *Conditional densities*

We give here a useful result for the density of $(X_1, \ldots, X_n)$.

Theorem 3.1 **(Density of n-th marginals)** *We define*

$$A_t = a + \sum_{j=t}^{\infty} a_j X_{t-j}, \; B_t = \sum_{j=t}^{\infty} b_j X_{t-j} \text{ and } C_i = A_i + \sum_{k=1}^{i-1} a_k x_{i-k},$$

with $C_1 = A_1$. Assume all variables and coefficients are non negative. If the ε_t are independent with marginal density f_{ε_t}, then $(X_1, \ldots, X_n)$ has a density $L(x_1, \ldots, x_n)$ conditionally to the past of the process. Forgetting the defi-

nition set, we have

$$L(x_1,\ldots,x_n) = \prod_{i=1}^{n} \frac{1}{|C_i|} f_{\varepsilon_i}\left(\frac{x_i - b_1 x_{i-1} - b_2 x_{i-2} - \cdots - b_{i-1}x_1 - B_i}{C_i}\right).$$

COROLLARY 3.2 *(Control of the density) Under the same assumptions as in theorem 3.1, if a is different from zero and if the densities of the (ε_t) are all bounded by M, then, for all $(x_1,\ldots,x_n)$,*

$$L(x_1,\ldots,x_n) \le \left(\frac{M}{a}\right)^n.$$

COROLLARY 3.3 *(Density of a couple) Under the same assumptions as in theorem 3.1, and if the (ε_t) are iid with density f, then the density p_i of the couple (X_1, X_i), satisfies $\|p_i\|_\infty \le \|f\|_\infty^2/A_1$ for all $i \in \mathbb{Z}$.*

Those lemmas are useful respectively for subsampling and functional estimation, see [2], [13] and [3]. For example, a standard kernel estimate of the density is classically proved to have variance $\sim f(x)\int K^2(s)ds/nh_n$ (with kernel function K, density f and bandwidth h_n) with corollary 3.3 and an additional dependence assumption $\theta_r = \mathcal{O}(r^{-a})$ for $a > 3$.

4 Proofs

We give in this section the proofs of theorem 2.1, theorem 2.2 and theorem 3.1. In the following, c denotes a constant that may vary from line to line.

4.1 *Proof of theorem 2.1*

We begin the proof by a useful lemma

LEMMA 4.1 *Assume that the coefficients (b_j) are non negative and $\|b\|_1 < 1$, then the coefficients (g_j) are non negative and $\|g\|_1 = (1 - \|b\|_1)^{-1}$.*

Proof of lemma 4.1. Since the (b_j) are non negative and $\|b\|_1 < 1$, a classical result shows that the development in power series $\sum_{j=0}^{+\infty} g_j z^j$ of the function $G(z) = \{1 - B(z)\}^{-1}$ has a radius bigger than 1. Moreover, after direct computations, we get the non negativity of the (g_j). Hence, as G is increasing on

$[0,1]$, we have for all positive integer n and for all $z \in [0,1]$

$$\sum_{j=0}^{n} g_j z^j \leq G(z) \leq G(1).$$

Consequently, $\sum_{j=0}^{+\infty} g_j \leq G(1)$. We conclude by a continuity argument.

We now prove theorem 2.1. We use the normal convergence in L^p of the series defined by (2), indeed

$$\begin{aligned}
\sum_{k=1}^{\infty} \sum_{s_k<\cdots<s_1\leq t} & \left\| g_{t-s_1} h_{s_1-s_2} \cdots h_{s_{k-1}-s_k} \ \varepsilon_{s_1} \cdots \varepsilon_{s_k} \right\|_{L^p} \\
& \leq \sum_{k=1}^{\infty} \sum_{s_k<\cdots<s_1\leq t} |g_{t-s_1} h_{s_1-s_2} \cdots h_{s_{k-1}-s_k}| \ \|\varepsilon_{s_1} \cdots \varepsilon_{s_k}\|_{L^p} \\
& \qquad \leq \|g\|_1 \sum_k \|h\|_1^{k-1} \|\varepsilon\|_{L^p}^k.
\end{aligned}$$

Hence $\|h\|_1 \|\varepsilon_1\|_{L^p} < 1$ implies $\|X_t\|_{L^p} < \infty$.

Since $H(z) = A(z)G(z)$, we derive $\|h\|_1 \leq \|a\|_1 \|g\|_1$. Non negativity of the (b_j) implies $g_i \geq 0$. Thus

$$\sup_{|z|<1} |G(z)| = \sup_{|z|<1} \Big| \sum_{i=0}^{\infty} g_i z^i \Big| = \sup_{|z|<1, z>0} \sum_{i=0}^{\infty} g_i z^i = \sum_{i=0}^{\infty} g_i = \|g\|_1.$$

Hence, $\|h\|_1 \leq \|a\|_1 \sup_{|z|<1} |1 - B(z)|^{-1} \leq \|a\|_1 \left(\inf_{|z|<1} |1 - B(z)| \right)^{-1}$. Now $\sum_j b_j < 1$ implies $\|h\|_1 \leq \|a\|_1 (1 - \sum_j b_j)^{-1} \leq \frac{\|a\|_1}{1 - \|b\|_1}$. Finally, if $\|a\|_1 \|\varepsilon_1\|_{L^p} + \|b\|_1 < 1$, then $\|\varepsilon\|_{L^p} \|h\|_1 < 1$.

4.2 *Proof of theorem 2.2*

Markovian case (a). We use the general Markov chain theory. Write $X_n = M(X_{n-1}, \ldots, X_{n-J}, \varepsilon_n)$. Then $Y_n = (X_n, \ldots, X_{n-J+1})$ is a Markov chain with $Y_n = F(Y_{n-1}, \varepsilon_n)$ where $x = (x_1, \ldots, x_J)$ and $F(x, \varepsilon) =$

$(M(x,\varepsilon), x_1, \dots, x_{J-1})$. Thus

$$\begin{aligned}\mathbb{E}|M(x,\varepsilon_1) - M(y,\varepsilon_1)| &\le \mathbb{E}\Big|\sum_{j=1}^{J}(a_j\varepsilon_1 + b_j)(y_j - x_j)\Big| \\ &\le \max_j |y_j - x_j| \ \mathbb{E}\Big(\sum_{j=1}^{J}|a_j\varepsilon_1 + b_j|\Big).\end{aligned}$$

We define $\|x\| = \max_{1\le j\le J} \eta^{j-1}|x_j|$. We have

$$\begin{aligned}\|F(x,\varepsilon_1) - F(y,\varepsilon_1)\| &\le \max\big\{|M(x,\varepsilon_1) - M(y,\varepsilon_1)| \ , \max_{1\le i<J} \eta^i|x_i - y_i|\big\} \\ \mathbb{E}\|F(x,\varepsilon_1) - F(y,\varepsilon_1)\| &\le \max\big\{\eta^J \max_{1\le i\le J}(|x_i - y_i|) \ , \max_{1\le i<J}(\eta^i|x_i - y_i|)\big\}.\end{aligned}$$

Finally,

$$\mathbb{E}\|F(x,\varepsilon_1) - F(y,\varepsilon_1)\| \le \eta\|x - y\|. \tag{3}$$

We then use the following lemma which is a vectorial extension of a result of Doukhan and Louhichi [12].

Lemma 4.2 *Assume $Y_n = (X_n, \dots, X_{n-J+1})$ is a Markov chain with $Y_n = F(Y_{n-1}, \varepsilon_n)$ and the (ε_n) are iid with $\mathbb{E}|\varepsilon_1| < +\infty$. Then, if equation* (3) *holds, $\theta_r = \mathcal{O}(\eta^r)$ as $r \to \infty$.*

Proof of lemma 4.2. For $f : \mathbb{R}^u \to \mathbb{R}$ with $\|f\|_\infty \le 1$, $g : \mathbb{R}^v \to \mathbb{R}$, $\operatorname{Lip} g < \infty$, $i_1 \le \dots \le i_u \le i_u + r \le n_1 \le \dots \le n_v$ and $i_u - i_1 > J$, we set $X_{\mathbf{i}} = (X_{i_1}, \dots, X_{i_u})$ and $X_{\mathbf{n}} = (X_{n_1}, \dots, X_{n_v})$. We have

$$\operatorname{Cov}[f(X_{\mathbf{i}}), g(X_{\mathbf{n}})] = \int f(x_{i_1}, \dots, x_{i_u})\{\mathbb{E}g(X_{\mathbf{n}}^{y^u}) - \mathbb{E}g(X_{\mathbf{n}})\}dP_{X_{i_1}, X_{i_1+1}\dots, X_{i_u}},$$

where $y^u = (x_{i_u}, x_{i_u-1}, \dots, x_{i_u-J+1})$ and $X_{\mathbf{n}}^{y_u}$ denotes the vector $X_{\mathbf{n}}$ knowing that $Y_{i_u} = y^u$. Now, it is clear that

$$\begin{aligned}|\mathbb{E}g(X_{\mathbf{n}}^{y^u}) - \mathbb{E}g(X_{\mathbf{n}})| &\le \int \mu(d\tilde{y}^u)\mathbb{E}|g(X_{\mathbf{n}}^{y^u}) - g(X_{\mathbf{n}}^{\tilde{y}^u})| \\ &\le \operatorname{Lip}(g)\sum_{z=1}^{v}\int \mu(d\tilde{y}^u)\mathbb{E}|X_{n_z}^{y^u} - X_{n_z}^{\tilde{y}^u}|.\end{aligned}$$

From inequality (3) and using that $\eta < 1$, we get

$$\mathbb{E}\|Y_{n_z}^{y^u} - Y_{n_z}^{\tilde{y}^u}\| \le \eta^{n_z - i_u}\|y^u - \tilde{y}^u\| \le \eta^r\|y^u - \tilde{y}^u\|$$

and consequently

$$\mathbb{E}|X_{n_z}^{y^u} - X_{n_z}^{\tilde{y}^u}| \le \eta^r\|y^u - \tilde{y}^u\|.$$

Finally, we get

$$\begin{aligned}
&\int dP_{X_{i_1},X_{i_1+1}\dots,X_{i_u}}\mu(d\tilde{y}^u)\mathbb{E}|X_{n_z}^{y^u} - X_{n_z}^{\tilde{y}^u}| \\
&\qquad \le \eta^r \int dP_{X_{i_1},X_{i_1+1}\dots,X_{i_u}}\mu(d\tilde{y}^u)\|y^u - \tilde{y}^u\| \\
&\qquad\qquad \le \eta^r \int dP_{X_{i_1},X_{i_1+1}\dots,X_{i_u}}\mu(d\tilde{y}^u)\sum_i |y_i^u - \tilde{y}_i^u| \le c\eta^r.
\end{aligned}$$

An explicit bound on θ_r follows. In the cases *(b)* and *(c)*, we shall need lemmas describing the behavior of the coefficients (g_j) and (h_j) involved by expansion (2). We derive this behavior from the decay rates of the initial parameters (a_j) and (b_j). The study of the coefficients (g_j) et (h_j) aims at controlling the tails of the coefficients series. We begin by some useful lemmas.

LEMMA 4.3 *Let* $\gamma_J = \max_{j>J}|g_j|$. *Then* $|g_k| \le \gamma_J\|b\|_1 + \|g\|_1\sum_{j=k-J}^{k}|b_j|$, $\forall k \in \mathbb{N}$, $\forall J \in \{1,\dots,k-1\}$.

Proof of lemma 4.3. By definition of G, $G(z)\{1 - B(z)\} = \sum_{k=0}^{\infty}\alpha_k z^k = 1$. Put $b_0 = -1$, then $1 - B(z) = -\sum_{j=0}^{\infty} b_j z^j$ thus $\alpha_k = \sum_{j=0}^{k} g_j b_{k-j}$. By identification : $\alpha_k = \delta_{0k}$. Thus we get recursive equations on the coefficients g_j : $\alpha_0 = -g_0 b_0 \Rightarrow\ g_0 = 1$. Then, for $k \ge 1$, $g_k = \sum_{j=0}^{k-1} g_j b_{k-j}$. For all $J \in \{1,\dots,k-1\}$,

$$\begin{aligned}
g_k &= \sum_{j=1}^{k-J-1} g_{k-j}b_j + \sum_{j=k-J}^{k} g_{k-j}b_j \\
&\le \gamma_J \sum_{j=1}^{k-J-1}|b_j| + \sum_{j=k-J}^{k}|g_{k-j}||b_j| \\
&\le \gamma_J\|b\|_1 + \|g\|_1\sum_{j=k-J}^{k}|b_j|.
\end{aligned}$$

We give now an explicit bound for the coefficients in the geometric case.

LEMMA 4.4 *(Control of γ_k under geometric decay)*
– *Geometric case (b) :* $\forall\ 1 < \beta_1 < \beta,\ \gamma_k = \mathcal{O}(\beta_1^{-k})$ *as* $k \to \infty$.

Proof of lemma 4.4. The series $\sum_{j=1}^{\infty} b_j r^j$ has a radius bigger than β. Thus, the development in power series of $\{1 - B(z)\}^{-1}$ has a radius of at least β. Consequently, for all $0 \le r < \beta$ the sequence $(g_j r^j)_j$ is bounded and so, there exists $M_r > 0$ such that $g_j \le M_r r^{-j}$ for all j.

We now turn to the tails of the series generated by (g_j) and (h_j).

LEMMA 4.5 *(Control of q_K thanks to r_K) Let*

$$r_K = \sum_{k=K+1}^{\infty} |g_k| \text{ and } q_K = \sum_{k=K+1}^{\infty} |h_k|.$$

For all $J < K$,

$$q_K \le \|g\|_1 \sum_{l \ge K+1-J} |a_l| + 2\ r_J\ \|a\|_1.$$

Proof of lemma 4.5. We have

$$\begin{aligned}
q_K &\le \sum_{k=K+1}^{\infty} \sum_{j=0}^{k} |a_{k-j} g_j| \\
&\le \sum_{j=0}^{\infty} |g_j| \sum_{k=\max(K+1,j+1)}^{\infty} |a_{k-j}| \\
&\le \sum_{j=0}^{K} |g_j| \sum_{k=K+1}^{\infty} |a_{k-j}| + \sum_{j=K+1}^{\infty} |g_j| \sum_{k=j+1}^{\infty} |a_{k-j}|.
\end{aligned}$$

We now control each term,

$$\sum_{j=K+1}^{\infty} |g_j| \sum_{k=j+1}^{\infty} |a_{k-j}| \le r_K \|a\|_1$$

and for all $J < K$,

$$\sum_{j=0}^{K} |g_j| \sum_{k=K+1}^{\infty} |a_{k-j}| = \sum_{j=0}^{J} |g_j| \sum_{k=K+1}^{\infty} |a_{k-j}| + \sum_{j=J+1}^{K} |g_j| \sum_{k=K+1}^{\infty} |a_{k-j}|$$
$$\leq \|g\|_1 \sum_{l \geq K+1-J} |a_l| + r_J \|a\|_1.$$

Lemma 4.6 *(Controls of r_K and q_K)*

– *Geometric case (b) : $r_K = \mathcal{O}(\beta_1^{-K})$ and $q_K = \mathcal{O}\big(\alpha^{\frac{-K \log \beta_1}{\log(\alpha/\beta_1)}}\big)$ as $K \to \infty$.*
– *Riemannian case (c) : $r_K = \mathcal{O}\big(K^{\frac{(1-\beta)\log \rho}{(1-\beta)\log 2 + \log \rho}}\big)$ and $q_K = \mathcal{O}\big(K^{\max(\frac{(1-\beta)\log \rho}{(1-\beta)\log 2+\log \rho}, -\alpha)}\big)$ as $K \to \infty$, with $\rho = \frac{1}{1+\frac{1-\|b\|_1}{B}} < 1$.*

Proof of lemma 4.6, (b). As $|g_j| \leq c\beta_1^{-j}$, we easily get

$$r_K = \mathcal{O}(\beta_1^{-K}) \text{ and } q_K \leq c(e^{(K-J)\log \alpha} + e^{-J \log \beta_1}).$$

We take $J = \lfloor \frac{K \log \alpha}{\log(\alpha/\beta_1)} \rfloor$and we obtain $q_K = \mathcal{O}\big(\alpha^{\frac{-K \log \beta_1}{\log(\alpha/\beta_1)}}\big)$ as $K \to \infty$.

Proof of lemma 4.6, (c). We have

$$\begin{aligned}
r_K &\leq \sum_{k=K+1}^{\infty} \sum_{j=0}^{k-1} |g_j b_{k-j}| \\
&\leq \sum_{j=0}^{\infty} |g_j| \sum_{k=\max(K+1,j+1)}^{\infty} |b_{k-j}| \\
&\leq \sum_{j=0}^{K} |g_j| \sum_{k=K+1}^{\infty} |b_{k-j}| + \sum_{j=K+1}^{\infty} |g_j| \sum_{k=j+1}^{\infty} |b_{k-j}| \\
&\leq \sum_{j=0}^{K} |g_j| \sum_{k=K+1-j}^{\infty} |b_k| + r_K \|b\|_1.
\end{aligned}$$

Thus, $r_K(1-\|b\|_1) \le \sum_{j=0}^{K} |g_j| \sum_{k=K+1-j}^{\infty} |b_k|$. Moreover,

$$\sum_{k=K+1-j}^{\infty} |b_k| \le \sum_{k=K+1-j}^{\infty} |b_k| k^{\beta} (K+1-j)^{-\beta} \le B(K-j)^{-\beta}.$$

Then, for all $J < K$,

$$r_K \frac{1-\|b\|_1}{B} \le \sum_{j=0}^{K} |g_j| (K-j)^{-\beta}.$$

Consequently,

$$\begin{aligned} r_K \frac{1-\|b\|_1}{B} &\le \sum_{j=0}^{J} |g_{K-j}| j^{-\beta} + \sum_{j=J+1}^{K} |g_{K-j}| j^{-\beta} \\ &\le r_{K-J} - r_K + \|g\|_1 \sum_{j=J+1}^{K} j^{-\beta} \\ &\le r_{K-J} - r_K + \|g\|_1 \frac{K^{1-\beta} - J^{1-\beta}}{1-\beta}. \end{aligned}$$

In particular, we get

$$r_{2K} \left(1 + \frac{1-\|b\|_1}{B}\right) \le r_K + \frac{\|g\|_1}{1-\beta} K^{1-\beta} (2^{1-\beta} - 1).$$

Thus we derive for all j the inequality

$$r_{2^{j+1}} \le \rho \; r_{2^j} + \gamma \; 2^{j(1-\beta)},$$

with $0 < \rho < 1$ and $\gamma > 0$. By induction,

$$0 \le r_{2^{j+1}} \le \sum_{k=0}^{j} \gamma \rho^k \; 2^{(1-\beta)(j-k)} \; + \rho^j \; \|g\|_1.$$

We now control the first term. We have

$$\begin{aligned}\sum_{k=0}^{j}\gamma\rho^k\ 2^{(j-k)\beta} &= \sum_{k=0}^{j}\gamma\rho^{j-k}2^{k(1-\beta)}\\ &= \sum_{k=0}^{J}\gamma\rho^{j-k}2^{k(1-\beta)} + \sum_{k=J+1}^{j}\gamma\rho^{j-k}2^{k(1-\beta)}\\ &\le \frac{\gamma}{1-2^{1-\beta}}(\rho^{j-J}+2^{J(1-\beta)}).\end{aligned}$$

We balance both terms putting $J = \lfloor\frac{j\log\rho}{(1-\beta)\log 2+\log\rho}\rfloor$ and we get

$$r_{2^j} = \mathcal{O}\big(2^{j\frac{(1-\beta)\log\rho}{(1-\beta)\log 2+\log\rho}}\big) \text{ as } j\to\infty.$$

Let K such that $2^j < K \le 2^{j+1}$, we have $0 \le r_K \le r_{2^j}$. Finally,

$$r_K = \mathcal{O}\big(K^{\frac{(1-\beta)\log\rho}{(1-\beta)\log 2+\log\rho}}\big), \text{ as } K\to\infty.$$

Using that for all $J < K$,

$$\sum_{l\ge K+1-J}|a_l| \le A(K-J)^{-\alpha},$$

taking $J = \lfloor K/2\rfloor$, we get $q_K = \mathcal{O}\big(K^{\frac{(1-\beta)\log\rho}{(1-\beta)\log 2+\log\rho}} + K^{-\alpha}\big)$ as $K\to\infty$, which concludes.

LEMMA 4.7 *(Bounding θ) For all $r, L,\ J>0$ such that $LJ<r$,*

$$\theta_r \le c(\tilde h^L + r_J + q_J).$$

Proof of lemma 4.7. The chaotic expansion (2) writes

$$X_t = a\sum_{l=1}^{\infty}\sum_{j_1=0}^{\infty}\sum_{j_2=1}^{\infty}\cdots\sum_{j_l=1}^{\infty} g_{j_1}h_{j_2}\cdots h_{j_l}\varepsilon_{t-j_1}\cdots\varepsilon_{t-(j_1+\cdots+j_l)}. \tag{4}$$

Consider the vectors $X_{\mathbf{i}} = (X_{i_1},\ldots,X_{i_u})$, and $X_{\mathbf{j}} = (X_{j_1},\ldots,X_{j_v})$ where $j_1 - i_u \ge r$. Take $f:\mathbb{R}^u\to\mathbb{R}$, $g:\mathbb{R}^v\to\mathbb{R}$ two functions such that $\mathrm{Lip}\,(g)<\infty$ and $\|f\|_\infty\le 1$. We define $X^*_{\mathbf{j}} = (X^*_{j_1},\ldots,X^*_{j_v})$, where X^*_t denotes the sums for which l stops at L and $j_1,\ldots,j_l$ stop at J in the expansion (4). Note that

$X^*_{\mathbf{j}}$ and $X_{\mathbf{i}}$ are independent if L and J satisfy $LJ < r$, thus

$$|\mathrm{Cov}[f(X_{\mathbf{i}}), g(X_{\mathbf{j}})]| \leq |\mathrm{Cov}[f(X_{\mathbf{i}}), g(X_{\mathbf{j}}) - g(X^*_{\mathbf{j}})]| + |\mathrm{Cov}[f(X_{\mathbf{i}}), g(X^*_{\mathbf{j}})]|.$$

The second term vanishes by independence. For simplicity, we forget the constant value a. We have

$$\begin{aligned}|\mathrm{Cov}[f(X_{\mathbf{i}}), g(X_{\mathbf{j}}) - g(X^*_{\mathbf{j}})]| &\leq 2\|f\|_\infty E|g(X_{\mathbf{j}}) - g(X^*_{\mathbf{j}})| \\ &\leq 2 \ \mathrm{Lip}\,(g)\|f\|_\infty \sum_{k=1}^{v} E|X_{j_k} - X^*_{j_k}| \\ &\leq 2v\mathrm{Lip}\,(g)\|f\|_\infty E|X_0 - X^*_0|,\end{aligned}$$

by the stationarity of (X_t, X^*_t). Thus, we may set

$$\theta_r = E|X_0 - X^*_{0,r,J,L}|,$$

where

$$X^*_{0,r,J,L} = \sum_{l=1}^{L} \sum_{j_1=0}^{J} \cdots \sum_{j_l=0}^{J} g_{j_1} h_{j_2} \cdots h_{j_l} \varepsilon_{-j_1} \cdots \varepsilon_{-(j_1+\cdots+j_l)}, \text{ if } JL < r.$$

Thus, we finally obtain

$$\begin{aligned}\theta_r &\leq \sum_{l=L+1}^{\infty} \|g\|_1 \tilde{h}^{l-1} \|\varepsilon\|_{L^1} + \sum_{l=1}^{\infty} r_J \tilde{h}^{l-1} \|\varepsilon\|_{L^1} + \sum_{l=1}^{\infty} q_J \|g\|_1 (l-1) \tilde{h}^{l-2} \|\varepsilon\|^2_{L^1} \\ &\leq c(\tilde{h}^L + r_J + q_J).\end{aligned}$$

End of the proof of the theorem 2.2. The end of the proof is dedicated to explicit this bound in terms of (a_j) and (b_j) decay rates.

- *Geometric case (b) :* $\theta_r = \mathcal{O}(\tilde{h}^L + \beta_1^{-J} + m^J) = \mathcal{O}(\tilde{h}^L + m^J)$ such that $JL < r$, as $J, L \to \infty$. Consequently,

$$\begin{aligned}\theta_r &= \mathcal{O}\big\{\big(e^{-\sqrt{L\log(\tilde{h})J\log(m)}}\big)\big(e^{-\sqrt{\frac{L\log(\tilde{h})}{J\log(m)}}} + e^{-\sqrt{\frac{J\log(m)}{L\log(\tilde{h})}}}\big)\big\} \\ &= \mathcal{O}\big(e^{-\sqrt{L\log(\tilde{h})J\log(m)}}\big) = \mathcal{O}\big(e^{-\sqrt{r\log(\tilde{h})\log(m)}}\big) \text{ as } r \to \infty.\end{aligned}$$

– *Riemannian case (c) :*

$$\theta_r = \mathcal{O}\Big\{\tilde{h}^L + J^{\max\left(\frac{(1-\beta)\log\rho}{(1-\beta)\log 2+\log\rho}, -\alpha\right)}\Big\},$$

where $J, L, r \to \infty$ in such a way that $JL < r$. We define

$$m'_{(\alpha,\beta,\rho)} = \max\Big(\frac{(1-\beta)\log\rho}{(1-\beta)\log 2 + \log\rho}, -\alpha\Big) \text{ and } C = \frac{m'_{(\alpha,\beta,\rho)}}{\log\tilde{h}}.$$

Take $L = \lfloor C\log J\rfloor$. We get $\theta_r = \mathcal{O}(J^{m'_{(\alpha,\beta,\rho)}})$. Consider now the largest possible integer $J = J_r$ such that $J\lfloor C\log J\rfloor < r$. Since $J_r \sim \frac{r}{C\log r}$, we finally obtain

$$\theta_r = \mathcal{O}\Big\{\Big(\frac{r}{\log r}\Big)^{m'_{(\alpha,\beta,\rho)}}\Big\}, \text{ as } r \to \infty.$$

4.3 *Proof of theorem 3.1*

We work conditional on the past of $\{X_s, s \le 0\}$. We set

$$M = \begin{pmatrix} 1 & -a_1\varepsilon_n - b_1 & -a_2\varepsilon_n - b_2 & \dots & -a_{n-1}\varepsilon_n - b_{n-1} \\ 0 & 1 & -a_1\varepsilon_{n-1} - b_1 & \dots & -a_{n-2}\varepsilon_{n-1} - b_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 1 & -a_1\varepsilon_2 - b_1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

then,

$$M\begin{pmatrix} X_n \\ X_{n-1} \\ \dots \\ X_1 \end{pmatrix} = \begin{pmatrix} A_n\varepsilon_n + B_n \\ A_{n-1}\varepsilon_{n-1} + B_{n-1} \\ \dots \\ A_1\varepsilon_1 + B_1 \end{pmatrix},$$

which can be written

$$\begin{cases} \varepsilon_1 = & \frac{X_1 - B_1}{A_1} \\ \varepsilon_2 = & \frac{X_2 - b_1X_1 - B_2}{a_1X_1 + A_2} \\ . & \\ . & \\ \varepsilon_n = & \frac{X_n - b_1X_{n-1} - b_2X_{n-2} - \cdots - b_{n-1}X_1 - B_n}{a_1X_{n-1} + a_2X_{n-2} + \cdots + a_{n-1}X_1 + A_n} \end{cases},$$

and we may set $(\varepsilon_1, \ldots, \varepsilon_n) = \phi(X_1, \ldots, X_n)$. Now

$$\mathbb{E}g(X_1, X_2, \ldots, X_n) = \int g\{\phi^{-1}(u_1, \ldots, u_n)\} f_{\varepsilon_1}(u_1) \cdots f_{\varepsilon_n}(u_n) du_1 \cdots du_n.$$

We put $(u_1, \ldots, u_n) = \phi(x_1, \ldots, x_n)$. The Jacobian is diagonal, hence we compute

$$\frac{\partial u_1}{\partial x_1} = C_1^{-1}, \ \frac{\partial u_2}{\partial x_2} = C_2^{-1}, \ldots, \frac{\partial u_n}{\partial x_n} = C_n^{-1}.$$

Proof of corollary 3.3. We prove the result for the density of the couple (X_1, X_4), we can prove the general result the same way. We have

$$\begin{aligned} p(x_1, x_4) &= \int L(x_1, \ldots, x_4) dx_2 dx_3 \\ &\leq \frac{\|f\|^2}{|C_1|} \int \frac{1}{|C_2 C_3|} f\Big(\frac{x_2 - b_1 x_1 - B_2}{C_2}\Big) f\Big(\frac{x_3 - b_1 x_2 - b_2 x_1 - B_3}{C_3}\Big) dx_2 dx_3. \end{aligned}$$

Hence we put : $u = C_2^{-1}(x_2 - b_1 x_1 - B_2)$ and $v = C_3^{-1}(x_3 - b_1 x_2 - b_2 x_1 - B_3)$. Direct computations give that the Jacobian matrix is diagonal and that its absolute value writes $|C_2 C_3|$, thus $p(x_1, x_4) \leq \frac{\|f\|^2}{A_1} \int f(u) f(v) du dv \leq \frac{\|f\|^2}{A_1}$.

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